

THE HESSE CURVE OF A LEFSCHTZ PENCIL OF PLANE CURVES

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ABSTRACT. We prove that for a generic Lefschetz pencil of plane curves of degree $d \geq 3$ there exists a curve H (called the Hesse curve of the pencil) of degree $6(d-1)$ and genus $3(4d^2 - 13d + 8) + 1$, and such that: (i) H has d^2 singular points of multiplicity three at the base points of the pencil and $3(d-1)^2$ ordinary nodes at the singular points of the degenerate members of the pencil; (ii) for each member of the pencil the intersection of H with this fibre consists of the inflection points of this member and the base points of the pencil.

1. Let $F(\bar{a}, \bar{z}) = \sum_{k+m+n=d} a_{k,m,n} z_1^k z_2^m z_3^n$ be the homogeneous polynomial of degree d in variables z_1, z_2, z_3 and of degree one in variables $a_{k,m,n}$, $k + m + n = d$. Denote by $\mathcal{C}_d \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$, where $K_d = \frac{d(d+3)}{2}$, the complete family of plane curves of degree d given by equation $F(\bar{a}, \bar{z}) = 0$. Let $\mathcal{I}_d = \mathcal{C}_d \cap \mathcal{H}_d$, where

$$\mathcal{H}_d = \{(\bar{a}, \bar{z}) \in \mathbb{P}^{K_d} \times \mathbb{P}^2 \mid \det\left(\frac{\partial^2 F(\bar{a}, \bar{z})}{\partial z_i \partial z_j}\right) = 0\}.$$

Denote by $f_d : \mathcal{C}_d \rightarrow \mathbb{P}^{K_d}$ the restrictions of the projection $\text{pr}_1 : \mathbb{P}^{K_d} \times \mathbb{P}^2 \rightarrow \mathbb{P}^{K_d}$ to \mathcal{C}_d . It is well-known (see, for example, [1]) that for a generic point $\bar{a}_0 \in \mathbb{P}^{K_d}$ the intersection of the curve $C_{\bar{a}_0} = f_d^{-1}(\bar{a}_0)$ and its Hessian curve $H_{C_{\bar{a}_0}}$ given by $\frac{\partial^2 F(\bar{a}_0, \bar{z})}{\partial z_i \partial z_j} = 0$ is the set of the inflection points of $C_{\bar{a}_0}$ containing $3d(d-2)$ points. Therefore for $d \geq 3$ the morphism $h_d = f_d|_{\mathcal{I}_d} : \mathcal{I}_d \rightarrow \mathbb{P}^{K_d}$ has degree $\deg h_d = 3d(d-2)$.

Let \mathcal{S}_d be a subvariety of \mathbb{P}^{K_d} consisting of the points \bar{a} such that the curves $C_{\bar{a}}$ are singular and let \mathcal{M}_d be a subvariety of \mathbb{P}^{K_d} consisting of the points \bar{a} such that for $\bar{a} \in \mathcal{M}_d$ the curve $C_{\bar{a}}$ has a r -tuple inflection point with $r \geq 2$. Let $\mathcal{B}_d = \mathcal{S}_d \cup \mathcal{M}_d$ (if $d = 3$ then $\mathcal{M}_3 = \emptyset$). It is easy to show ([5]) that \mathcal{M}_d is an irreducible hypersurface in \mathbb{P}^{K_d} if $d \geq 4$. It is well-known also that \mathcal{S}_d is an irreducible hypersurface in \mathbb{P}^{K_d} , $\deg \mathcal{S}_d = 3(d-1)^2$.

Proposition 1. ([5]) *The local monodromy group¹ of h_d at a generic point $\bar{a} \in \mathcal{M}_d$ is a subgroup \mathbb{Z}_2 of the symmetric group $\mathbb{S}_{3d(d-2)}$ generated by a transposition, and the local monodromy group at a generic point $\bar{a} \in \mathcal{S}_d$ is a subgroup $\mathbb{Z}_3 \subset \mathbb{S}_{3d(d-2)}$ generated by the product of two disjoint cycles of length three.*

¹The definition of the local monodromy group of a dominant morphism $\varphi : X \rightarrow \mathbb{P}^m$ at a point $p \in \mathbb{P}^m$ can be found, for example, in [3] or [4].

2. Remind that, by definition, a *Lefschetz pencil* is a fibration $f_d : C_L = f_d^{-1}(L) \rightarrow L$ of curves over L (and also the linear system $\{C_{\bar{a}}\}_{\bar{a} \in L}$ of curves of degree d in \mathbb{P}^2), where L is a line in \mathbb{P}^{K_d} in general position with respect to the divisor \mathcal{S}_d . The body C_L of the Lefschetz pencil is a non-singular surface. The linear system $\{C_{\bar{a}}\}_{\bar{a} \in L}$ has d^2 base points and the restriction of the projection pr_2 to C_L is the composition of d^2 σ -processes with centers at the base points. We say that the Lefschetz pencil $f_d : C_L \rightarrow L$ is *generic* if L and $\mathcal{M}_d \setminus \mathcal{S}_d$ meet at $m_d = \deg \mathcal{M}_d$ different points.

Proposition 2. *We have $\deg \mathcal{M}_d = 6(d-3)(3d-2)$.*

Proof. Consider a generic Lefschetz pencil $\{C_{\bar{a}}\}_{\bar{a} \in L}$. It follows from Theorem 1 in [5] that the curve $I_L = h_d^{-1}(L) = C_L \cap H_{C_L}$ is irreducible. Also, it is easy to show ([5]) that the curve I_L has $\deg S_d = 3(d-1)^2$ singular points which are the ordinary nodes. Let $\nu : \tilde{I}_L \rightarrow I_L$ be the normalization of I_L , g_d the genus of \tilde{I}_L , and $\tilde{h}_d = h_d \circ \nu : \tilde{I}_L \rightarrow L$. It follows from Proposition 1 that $\tilde{h}_d : \tilde{I}_L \rightarrow L$ is ramified at m_d points with multiplicity two (at the preimages of 2-tuple inflection points of the fibres of the Lefschetz pencil) and it is ramified at $2 \deg S_d = 6(d-1)^2$ points with multiplicity three (at the preimages of the singular points of I_L). Therefore, it follows from Hurwitz formula that

$$2(g_d - 1) = -2 \deg \tilde{h}_d + m_d + 4 \deg S_d = -6d(d-2) + m_d + 12(d-1)^2. \quad (1)$$

On the other hand, the curve $I_L \subset L \times \mathbb{P}^2$ is a complete intersection of H_{C_L} and the smooth surface C_L . The Picard group $Pic(L \times \mathbb{P}^2)$ is a free abelian group generated by divisors $A = pr_1^{-1}(t)$ and $B = pr_2^{-1}(\mathbb{P}^1)$, where t is a point in L and \mathbb{P}^1 is a line in \mathbb{P}^2 . The canonical class $K_{L \times \mathbb{P}^2} = -2A - 3B$, $C_L \in |A + dB|$, and $H_{C_L} \in |3A + 3(d-2)B|$. Let us restrict the divisors A and B to the surface C_L . It follows from the adjunction formula that $K_{C_L} = -A + (d-3)B$. Besides, we have $I_L \in |3A + 3(d-2)B|$, $(A, A)_{C_L} = (A, A, A + dB)_{L \times \mathbb{P}^2} = 0$, $(A, B)_{C_L} = (A, B, A + dB)_{L \times \mathbb{P}^2} = d$, and $(B, B)_{C_L} = (B, B, A + dB)_{L \times \mathbb{P}^2} = 1$.

Therefore we have $2(g_d - 1) = (I_L, I_L + K_{C_L})_{C_L} - 2 \deg S_d = 6(4d^2 - 13d + 8)$. Comparing this equality with (1), we obtain that $m_d = 6(d-3)(3d-2)$. \square

Let L be a line in \mathbb{P}^{K_d} , $L \not\subset \mathcal{S}_d$, and $I_L = h_d^{-1}(L)$. We call $H_L = pr_2(I_L) \subset \mathbb{P}^2$ the *Hesse curve of the pencil* $\{C_{\bar{a}}\}_{\bar{a} \in L}$.

Since for a generic Lefschetz pencil $\{C_{\bar{a}}\}_{\bar{a} \in L}$ the Hesse curve H_L is irreducible, simple calculations show that the number of elements of a generic Lefschetz pencil having an inflection point at a base point of the pencil is less than or equal to three. Besides, we have $\deg H_L = (H_L, \mathbb{P}^1)_{\mathbb{P}^2} = (I_L, B)_{C_L} = 6(d-1)$. Therefore we have

Theorem 3. *The Hesse curve H_L of a generic Lefschetz pencil $\{C_{\bar{a}}\}_{\bar{a} \in L}$ of plane curves of degree $d \geq 3$ has the following properties:*

- (i) $\deg H_L = 6(d-1)$ and its genus is equal to $3(4d^2 - 13d + 8) + 1$;

- (ii) H_L has d^2 singular points of multiplicity three at the base points of the pencil $\{C_{\bar{a}}\}_{\bar{a} \in L}$ and $3(d-1)^2$ ordinary nodes at the singular points of the degenerate fibres of the pencil;
- (iii) for each $\bar{a} \in L$ the intersection $H_L \cap C_{\bar{a}}$ consists of the inflection points of $C_{\bar{a}}$ and the base points of the pencil, and if $p \in H_L \cap C_{\bar{a}}$ is the 2-tuple inflection point of $C_{\bar{a}}$, then H_L and $C_{\bar{a}}$ touch each other at p .

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